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FREE OSCILLATIONS OF LIQUID IN RIGID VESSELS

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Small steady oscillations of a perfect incompressible liquid in a rigid vessel are examined. Although this problem was fairly thoroughly investigated [1 - 3], the determination of high frequency oscillations and of their form in a liquid in vessels of arbitrary shape presents considerable difficulties.
A simplified approximate method, whose accuracy increases at higher frequencies is proposed for solving this problem. It is shown on the example of several problems that for practical purposes this method can be used for the full range of frequencies. Estimates of the lower and upper bounds are given in some of the problems.

1. The velocity potential Φ of free oscillations of liquid satisfies the Laplace equation with boundary conditions [4]

$$\Delta\Phi = 0 \quad \text{in } V$$

$$\frac{\partial\Phi}{\partial z} - \frac{\lambda^2}{R}\Phi = 0 \quad \left(\lambda^2 = \frac{\omega^2 R}{g}\right) \text{ along } \Sigma, \quad \frac{\partial\Phi}{\partial n} = 0 \text{ along } S \qquad (1.1)$$

where S is the wetted part of the vessel surface, Σ is the free surface of the unperturbed liquid. V is the region bounded by the surface $S + \Sigma$, $\partial\Phi / \partial n$ is a derivative along the normal to S, R is a constant of dimension length (a characteristic dimension of the cavity), ω is the angular oscillation frequency, g is the acceleration of gravity, and the direction of the Oz-axis is opposite to that of the gravity force vector (Fig. 1).

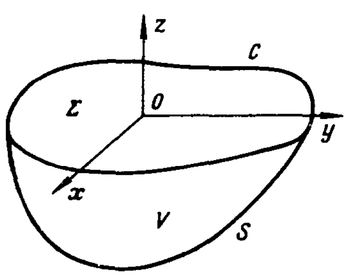

Fig. 1

Let us establish a certain property of function Φ for $\lambda \to \infty$. Assuming that functions in Green's formula are equal to Φ, with the use of (1.1) we obtain

$$\iiint\limits_V \left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] dV = \frac{R}{\lambda^2}\iint\limits_\Sigma \left(\frac{\partial\Phi}{\partial z}\right)^2 d\Sigma$$

Hence

$$\iiint\limits_V \left(\frac{\partial\Phi}{\partial z}\right)^2 dV \leqslant \frac{R}{\lambda^2} \iint\limits_\Sigma \left(\frac{\partial\Phi}{\partial z}\right)^2 d\Sigma \qquad (1.2)$$

Let $\lambda \to \infty$. It then follows from (1.2) that it is not possible to separate in V a finite volume in which the order of function $\partial\Phi / \partial z$ would not be less than the order of this function at Σ. Hence we write

$$\max\left|\frac{\partial\Phi}{\partial z}\right|_{z<-\delta} \ll \max\left|\frac{\partial\Phi}{\partial z}\right|_{z=0} \qquad (1.3)$$

where δ is an arbitrarily small positive number. A similar reasoning yields the strong inequalities

$$\max\left|\frac{\partial\Phi}{\partial x}\right|_{z<-\delta} \ll \max\left|\frac{\partial\Phi}{\partial z}\right|_{z=0} \qquad (1.4)$$

$$\max\left|\frac{\partial\Phi}{\partial y}\right|_{z<-\delta} \ll \max\left|\frac{\partial\Phi}{\partial z}\right|_{z=0}$$

We note that (1.3) and (1.4) are consistent with the known solutions of particular problems, which show that the free oscillation amplitude of liquid particles is attenuated with increasing distance from the liquid free surface. This attenuation is intensified with increasing oscillation frequency. For considerable values of λ the second of conditions (1.1) with (1.3) and (1.4) taken into consideration can be approximated by condition

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{along } C \qquad (1.5)$$

where C is the boundary of surface Σ.

Let γ_0 be the angle between the outward normal to S at points of boundary C and the Oz-axis, and let $\partial\Phi / \partial\nu_0$ be a derivative along the outward normal to C (in the Σ-plane). Condition (1.5) can now be written as

$$\frac{\partial\Phi}{\partial\nu_0}\sin\gamma_0 + \frac{\partial\Phi}{\partial z}\cos\gamma_0 = 0 \quad \text{along } C \qquad (1.6)$$

Using the method of separation of variables, we obtain the harmonic function $\partial\Phi / \partial z$ in the form

$$\frac{\partial\Phi}{\partial z} = \sum_{i=1}^{\infty} \varkappa_i \varphi_i(x, y)\, e^{\varkappa_i z} \qquad (\varkappa_i > 0)$$

(in accordance with (1.3) we discard the particular solutions which show no attenuation with increasing distance from the liquid free surface). From this we find

$$\Phi = \psi(x, y) + \sum_{i=1}^{\infty} \varphi_i(x, y)\, e^{\varkappa_i z} \qquad (1.7)$$

Substituting this expansion into the input Laplace equation and (1.6), we obtain

$$\Delta\varphi + \varkappa^2\varphi = 0 \qquad (1.8)$$

$$\frac{\partial\varphi}{\partial\nu_0}\operatorname{tg}\gamma_0 + \varkappa\varphi = 0 \quad \text{along } C \qquad (1.9)$$

Here and in the following subscript i is omitted. The relationship between $\varkappa$ and the dimensionless parameter λ of oscillation frequency, derived by substituting (1.7) into the first of conditions (1.1), is

$$\lambda^2 = \varkappa R \qquad (1.10)$$

The approximate solution of the three-dimensional input problem (1.1) of determining the eigenvalues λ^2 / R and eigenfunctions Φ (x, y, z) can thus be derived for high λ by solving the considerably simpler two-dimensional problem (1.8), (1.9) of finding the eigenvalues $\varkappa^2$ and eigenfunctions $\varphi(x, y)$.

The following important conclusion can be reached on the basis of (1.1) – (1.7): the approximate solution of the considered problem depends at sufficiently high oscillation frequencies only on the dimensions and configuration of the liquid free surface boundary C in its unperturbed state and on the angle γ_0 between the Oz -axis and the outward normals to the vessel wetted surface S drawn through points of that boundary, and is independent of other geometric dimensions of the vessel.

2. Let us define certain properties of the spectrum of eigenvalues of problem (1.8), (1.9). We limit the analysis to that part of the spectrum which corresponds to condition $\varkappa > 0$ (we recall that it is precisely these solutions that are of interest). We note that the problem (1.8), (1.9) is unusual in that the parameter appears in both the equation and the boundary condition.

Let us prove the following theorem: The eigenvalues $\varkappa^2$ of problem (1.8), (1.9) con stitute an infinitely increasing sequence

$$\lim_{k\to\infty} \varkappa_k^2 = \infty \qquad (2.1)$$

The eigenvalues λ^2/R of problem (1.1) are known to posess this property [2, 5].

To prove the theorem we substitute in (1.9) for parameter $\varkappa$ a certain positive arbitrary number σ such that

$$\frac{\partial\varphi}{\partial\nu_0} + \sigma\varphi \operatorname{ctg}\gamma_0 = 0 \quad \text{along } C \qquad (2.2)$$

For any σ the eigenvalues $\varkappa^2(\sigma)$ of problem (1.8), (2.2) are real and constitute a sequence in which [6]

$$\lim_{k\to\infty} \varkappa_k^2(\sigma) = \infty$$

It is shown in [6] that each of the $\varkappa_k^2(0)$ becomes $\varkappa_k^2(\infty)$, continuously varying with the continuous variation of σ from zero to infinity, and

$$\varkappa_1^2(0) = 0, \quad \varkappa_k^2(0) > 0 \quad (k = 2, 3, \ldots); \quad \varkappa_k^2(\infty) > 0 \quad (k = 1, 2, 3, \ldots)$$

Hence for every k, except possibly $k = 1$, there exists at least one σ for which the condition $\varkappa_k^2(\sigma) = \sigma^2$ is satisfied. It is this condition which determines the eigenvalues $\varkappa_k^2$ of problem (1.8), (1.9). Since the asymptotic behavior of the kth eigenvalue is independent of the boundary condition (is independent of σ) [6], hence (2.1) is valid. We point out that with the decrease of angle γ_0 every eigenvalue of problem (1.8), (1.9) along boundary C can only increase.

3. The approximate solution of problem (1.1) can be obtained by solving (1.8), (1.9) if the oscillation frequencies are sufficiently high. It is shown below on the example of actual solutions of a few particular problems that in many practical calculations it is possible to consider all unknown frequencies without exception, as being sufficiently high.

1. A cylindrical vessel with a flat bottom perpendicular to the generatrix of its side surface (Fig. 2). In this problem, which was considered in [3, 7, 8], the velocity

potential of the motions of liquid particles and the frequency parameters are determined by formulas

$$\Phi = \varphi(x, y) \operatorname{ch} \varkappa (z + h), \qquad \lambda^2 = \varkappa R \operatorname{th} \varkappa h \tag{3.1}$$

where $\varkappa$ and $\varphi(x, y)$ are, respectively, the eigenvalues and the eigenfunctions of the problem as follows:

$$\Delta\varphi + \varkappa^2\varphi = 0, \qquad \partial\varphi / \partial n = 0 \text{ along } C \tag{3.2}$$

Solving this problem by the proposed approximate method for $\gamma_0 = \pi / 2$, from (1.8), (1.9) we obtain (3.2), and instead of (3.1) we have

$$\Phi = \varphi(x, y) e^{\varkappa z}, \qquad \lambda^2 = \varkappa R \tag{3.3}$$

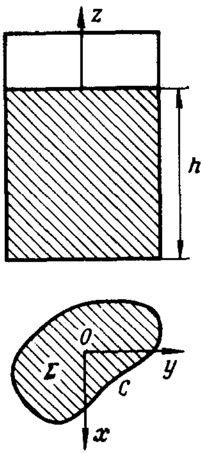

Fig. 2

Evidently (3.1) can be approximated by formulas (3.3) provided that $\varkappa h$ is sufficiently great. Hence, as expected the accuracy of the approximate method increases with increasing depth of liquid and oscillation frequency. In the case of a circular cylindrical cavity of radius R the accuracy of the determination of the first two frequency parameters λ_1 and λ_2 by (3.3) is within 5%, if h / R is equal 0.81 and 0.49, respectively.

2. An infinitely long circular channel with a horizontal axis partly filled with liquid. This problem was reduced in [9] to an integral equation subsequently solved numerically on a computer. The effect of the depth of liquid on parameters of the first three oscillation frequencies and on the shape of the liquid free surface at these frequencies is shown there in the form of curves for $e = 0$ $(\gamma_0 = \pi / 2)$. The solution relates to oscillations antisymmetric with respect to the vertical plane $y = 0$ drawn through the vessel axis.

In this case $\varphi = \varphi(y)$, and from (1.8), (1.9) we obtain

$$d^2\varphi/dy^2 + \varkappa^2\varphi = 0$$

$$\pm \frac{d\varphi}{dy} \operatorname{tg} \gamma_0 + \varkappa\varphi = 0 \qquad \text{for} \quad y = \pm R \sin \gamma_0$$

From this we conclude that for antisymmetric oscillations with respect to plane $y = 0$ the solution is of the form

$$\varphi_k = \sin \lambda_k{}^2\psi, \qquad \lambda_k{}^2 = \frac{k\pi - \gamma_0}{\sin \gamma_0} \qquad \left(\psi = \frac{y}{R}, \ k = 1, 2, \ldots\right) \tag{3.4}$$

The results presented in [9] very closely agree with those derived from solution (3.4), while the curves showing the effect of depth of liquid on the third frequency parameter and on the free surface form for all three oscillation frequencies throughout the depth of liquid are, within drawing accuracy, exactly the same.

3. An arbitrary cavity filled with liquid whose free surface has the form of a circular ring. Let us write (1.8), (1.9) in a polar system of coordinates

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2\varphi}{\partial\theta^2} + \varkappa^2\varphi = 0 \tag{3.5}$$

$$\frac{\partial\varphi}{\partial r} \operatorname{tg} \gamma_0 + \varkappa\varphi = 0 \qquad \text{for } r = R_1, \ r = R_2 \tag{3.6}$$

where R_1 and R_2 are, respectively, the radii of the inner and outer contours of the circular ring of the unperturbed liquid free surface.

Setting

$$\varphi_n = \chi_n(\rho) \cos n\theta \qquad (\rho = \varkappa r) \tag{3.7}$$

and taking into account (3.5), we obtain

$$\rho^2 \frac{d^2\chi_n}{d\rho^2} + \rho \frac{d\chi_n}{d\rho} + (\rho^2 - n^2)\chi_n = 0$$

The general solution of this equation is

$$\chi_n = C_1 J_n(\rho) + C_2 Y_n(\rho) \tag{3.8}$$

where $J_n(\rho)$ and $Y_n(\rho)$ are Bessel functions of the nth order of the first and second kind, respectively.

Let us now consider the particular case in which for $r = R_1$ and $r = R_2$ the value of γ_0 becomes equal to γ_1 and γ_2 respectively. The boundary conditions (3.6) then assume the form

$$\frac{d\chi_n}{d\rho} \operatorname{tg}\gamma_0 + \chi_n = 0 \qquad \text{for } \rho = \varkappa R_1, \quad \rho = \varkappa R_2 \tag{3.9}$$

Substituting (3.8) into (3.9), we obtain

$$a_{11}C_1 + a_{12}C_2 = 0, \qquad a_{21}C_1 + a_{22}C_2 = 0 \tag{3.10}$$

$$a_{11} = a_{11}(\gamma_1, R_1) = \left[J_{n-1}(\varkappa R_1) - \frac{n}{\varkappa R_1} J_n(\varkappa R_1)\right] \operatorname{tg}\gamma_1 + J_n(\varkappa R_1)$$

$$a_{12} = a_{12}(\gamma_1, R_1) = \left[Y_{n-1}(\varkappa R_1) - \frac{n}{\varkappa R_1} Y_n(\varkappa R_1)\right] \operatorname{tg}\gamma_1 + Y_n(\varkappa R_1)$$

$$a_{21} = a_{11}(\gamma_2, R_2), \qquad a_{22} = a_{12}(\gamma_2, R_2)$$

The condition for the determinan of system (3.10) to vanish yields the following equation for the determination of the oscillation frequency:

$$a_{11}a_{22} - a_{12}a_{21} = 0 \tag{3.11}$$

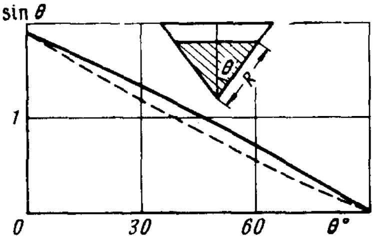

Fig. 3

In the particular case of a circular free surface of the unperturbed liquid ($R_1 = C_2 = 0$, $R_2 = r_0$, $\gamma_2 = \gamma_0$ and $C_1 = 1$) (3.8) and (3.11) reduce to the much simpler form

$$\chi_n = J_n(\rho) \tag{3.12}$$

$$\left[J_{n-1}(\varkappa r_0) - \frac{n}{\varkappa r_0} J_n(\varkappa r_0)\right] \operatorname{tg}\gamma_0 + J_n(\varkappa r_0) = 0 \tag{3.13}$$

a) A circular conical vessel. This problem is solved in [10] on a computer with the use of the method of variations. The calculation results for the parameter of the first natural frequency are shown in Fig. 3 (solid line) together with data obtained with the use of (3.13), shown by the dotted line.

b) A spherical vessel. The solution of this problem derived by the method described above for the case of an infinitely long circular channel is given in [9]. The results for the parameters of the first three frequencies λ_1, λ_2 and λ_3 for $n = 1$ are shown in Fig. 4 (solid lines). The shape of the liquid free surface corresponding to these oscillation frequencies for $\varepsilon = 0$ ($\gamma_0 = \pi/2$) is given in Fig. 5.

For comparison curves calculated by Eq. (3.13) are shown in Fig. 4 by dotted lines. The shape of the liquid free surface determined by (3.12), shown in Fig. 5, is in agreement with that obtained in [9].

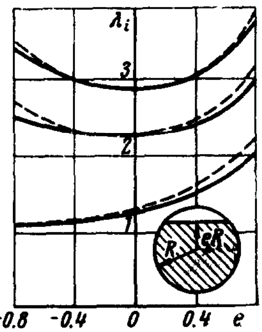

Fig. 4

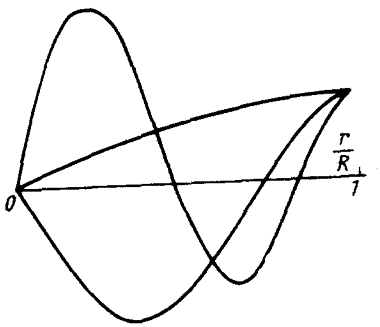

Fig. 5

4. The approximate solution derived here by the proposed method can be considered as exact in the case of a certain cavity whose Σ and γ_0 match those of the investigated vessel and the angle γ between the outward normal to S and the Oz-axis satisfy condition

$$\frac{\partial \varphi}{\partial \nu} \operatorname{tg} \gamma + \varkappa \varphi = 0 \quad \text{along } S \tag{4.1}$$

This condition in which $\partial\varphi / \partial\nu$ is a derivative in the direction of the outer normal to the line of intersection of surface S and the plane $z = \text{const}$ (in a plane parallel to Σ) was obtained by analogy to (1.9).

Let us now consider the particular case of circular free surface of the liquid and $\gamma_0 = \text{const}$. Taking into consideration (3.7) and (3.12) and also that $\operatorname{tg} \gamma = -\partial z / \partial r$ from (4.1) we obtain

$$\zeta = \int_1^{\xi} \frac{J_n(\rho_0 \xi)\, d\xi}{J_{n-1}(\rho \xi) - n(\rho_0 \xi)^{-1} J_n(\rho_0 \xi)}, \quad \rho_0 = \varkappa r_0, \quad \xi = \frac{r}{r_0}, \quad \zeta = \frac{z}{r_0} \tag{4.2}$$

where r_0 is the radius of the liquid free surface.

Thus the approximate solution of the problem of free oscillations of liquid in a rigid vessel with a circular free surface is, for $\gamma_0 = \text{const}$ the exact solution of a similar problem of a cavity whose surface is a surface of rotation (4.2). In some cases it is possible to derive with the use of formula (4.2) fair estimates of the lower and upper bounds of free oscillation frequencies of a liquid with circular free surface in a vessel of arbitrary form.

Let us, for example, find such estimates for the first-order frequency in the case of a spherical cavity with $\gamma_0 = 135°$ and $n = 1$. For $\gamma_0 = 135°$ and $\gamma_0 = 139°$ the dimensionless frequency parameters are, respectively, $\lambda = 1.08$ and $\lambda = 1.02$. Using these data ($\rho_0 = 0.824$ and $\rho_0 = 0.743$) we construct two cavities (4.2) and find that for $\gamma_0 = 139°$ the cavity is wholly contained in the spherical vessel, while for $\gamma_0 = 135°$ (with equal free surfaces) the latter fits into the cavity. Hence by virtue of the known theorem it is possible to state that $1.02 < \lambda < 1.08$, where λ is the sought frequency parameter. Taking $\lambda = 1.05$ results in an error in the determination of this parameter not exceeding 3%.

The concept of comparing the sought frequency with that of free oscillations of liquid in vessels for which exact solutions are known (circular cylindrical cavities) was used in [11] for determining the lower and upper estimates. The obtained estimates, however apply only to cavities having surfaces of rotation of a particular form, viz., $r = r_0$ for $-z \leqslant h_0$ $(h_0 > 0)$ and $r \leqslant r_0$ for $-z > h_0$.

If one is interested only in finding whether the derived frequencies of free oscillations of liquid in a certain rigid vessel (without abrupt changes of curvature in its meridian section) are above or below their true values, it is sufficient to compare the curvatures in the meridian section of a particular vessel with that defined by (4.2) for the same γ_0 at $z = 0$. If the curvature of the investigated vessel is greater (smaller) than that of the vessel defined by (4.2), the obtained frequency is to be considered as an upper (lower) estimate.

The curvature K of the meridian section of the vessel defined by (4.2) can be calculated by formula

$$r_0 K = \left(1 + \frac{n^2 \operatorname{tg} \gamma_0}{\varkappa r_0} - \frac{\varkappa r_0}{\sin \gamma_0 \cos \gamma_0}\right) \sin \gamma_0 \cos^2 \gamma_0 \tag{4.3}$$

which is obtained by way of comparing Eq. (4.1), differentiated with respect to the meridian arc, with the Laplace equation (3.5) in which function φ has been eliminated by using (4.1).

Let us, for example, calculate the curvature (4.3) for $\gamma_0 = 135^\circ$. We find it to be greater than the curvature of the meridian section of a cone and smaller than that of a sphere. Hence the calculated frequency for the sphere is excessive, while that for the cone is deficient. This conclusion fonforms to the curves in Figs. 3 and 4.

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NOTES ON THE THREE-DIMENSIONAL FLOW PATTERN OF A PERFECT FLUID IN THE PRESENCE OF A SMALL PERTURBATION OF THE INITIAL VELOCITY FIELD

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Euler equations of the three-dimensional motion of a perfect incompressible fluid, linearized for a nearly stationary flow are considered and the class of stationary flows for which these linearized equations admit exact explicit solutions is indicated. The analysis of derived equations shows that in some stationary flows the perturbation buildup considerably differs from that obtaining in cases generally considered in the theory of hydrodynamic stability: there appears an infinitely great number of unstable configurations, the flow pattern is difficult to predict (since an approximate determination of perturbation development with time necessitates a rapidly increasing amount of information about initial conditions, etc). These differences are due to the different geometry of stationary flows. In the recently constructed models of stationary flows the assumption is made that a fluid particle in motion stretches into a filament or ribbon whose length exponentially increases with time, while in the usually considered flows the length is assumed to be a linear function of time. In two-dimensional flows the phenomenon of exponential stretching of particles is impossible. It is shown that this is, also, impossible in three-dimensional flows in which the vectors of velocity and viscosity are not collinear.

1. The linearized Euler equation. The shortened equation.

Let us write Euler's equation in the form of a vortex equation

$$\partial \mathbf{r} / \partial t = \{\mathbf{v}, \mathbf{r}\} \qquad (\mathbf{r} = \operatorname{rot} \mathbf{v}) \tag{1.1}$$

where the Poisson's bracket of the two vector fields is defined by the condition

$$D_{\{\mathbf{a},\mathbf{b}\}} = D_{\mathbf{b}} D_{\mathbf{a}} - D_{\mathbf{a}} D_{\mathbf{b}}$$

in which $D_{\mathbf{q}}$ denotes integration in the direction of field $\mathbf{q}$. Let us consider a small perturbation $\mathbf{u}$ of the stationary flow $\mathbf{v}$. Let $\mathbf{s}$ be the vortex perturbation field: $\operatorname{rot}(\mathbf{v} + \mathbf{u}) = \mathbf{r} + \mathbf{s}$. Equation (1.1) linearized in the neighborhood of flow $\mathbf{v}$ is of the form

$$\partial \mathbf{s} / \partial t = \{\mathbf{v}, \mathbf{s}\} + \{\operatorname{rot}^{-1} \mathbf{s}, \mathbf{r}\} \tag{1.2}$$

The operation rot^{-1} is understood as the restitution of a nondivergent vector field over